Intro Lecture on Tuesday, January 27, 2015

I. Give the course URL and instruct students to read it carefully and thoroughly, and, if they have questions, to raise them in lecture or recitation or send me/GAs an email:
   http://www2.math.umd.edu/~rlipsman/courses/math246.s15/
   They can also get to it off the Canvas course page. The link to the online course (NODE) Notes are there as well:
   https://courses.math.umd.edu/math246/NODE/1415S/main.html

II. The subject matter of the course is *ODE* = Ordinary Differential Equations.
   a. In this course we will attempt to answer the following questions concerning ODEs: What are they? How do they arise? Can we solve them? What do the solutions tell us? What do we do when we can't solve them?
   b. The framework in which we seek the answers is that of *Mathematical Modeling*. In virtually all sciences, i.e., physical, chemical, social, as well as in engineering, finance, game theory and other areas, we seek to solve problems by building a mathematical model for them – usually one or more equations, the solution(s) to which provide answers to the original problems. We compare those answers against real world measurements, refine the model if necessary and repeat the process until we get to a solution that matches with real world observations. The point is that **very** often, the equations in our models involve rates of change of the quantities we measure, that is, derivatives, thus *differential equations*. The word *ordinary* refers to the fact that the models we investigate will only involve one independent variable.

III. Examples:
   a. Bacterial growth in a medium, radioactive decay or continuous compound interest, $y'=ry$.
   b. Falling body with air resistance, $F = ma = m(dv/dt) = -mg + cv$.
   c. Rigid pendulum, $d^2\Theta/dt^2 + (g/L) \sin \Theta = 0$.

IV. Specific Features of ODEs
   a. Order of an ODE
   b. Explicit $y^{(n)} = f(t, y, \ldots, y^{(n-1)})$ or implicit $F(t, y, \ldots, y^{(n)}) = 0$.
   c. Linear vs non-linear
   d. Single equation vs system of eqns.

V. Techniques we will employ to study ODEs.
   a. **Algebraic or Symbolic**. This is where by algebra/calculus techniques we find a formula that represents a solution to the differential equation. The three main techniques that we will develop for 1st order equations are: *linear, separable* and *exact*. Give as an example $y' = ay + b$, a and b fixed constants. (This equation is both linear and separable, but not exact.) If $a = 0$, then $y = bt + c$ is a soln for any constant c. If a is not 0, then the solution is

   $$y = (1/a)(k \exp(at) - b)$$

   where k is any constant. We will derive this soln formula soon, but for now just verify that the formula given solves the eqn. Draw a few soln curves; note that they don't intersect. Mention that later we will formalize this observation with the so-called *Existence and Uniqueness Theorem.*

We call a soln formula with the arb constant a *general solution*. To get a specific soln, we must have *initial data*, e.g., $y(0) = 1$. Then we can determine the arb constant. Mention that an ODE with initial data is called an *initial value problem* (IVP).

To clarify: A ***solution*** to a first order ODE $y' = f(t, y)$, is a function $y = \phi(t)$, defined on some interval $a < t < b$, so that for every t in the interval, $\phi'(t) = f(\phi(t), t)$. A solution to an IVP: $y' = f(t, y)$, $y(t_0) = y_0$, is a solution defined on an interval containing t_0, so that $\phi(t_0) = y_0$.

b. **Geometric** or **Graphical**. We draw the *direction field* of the ODE $y' = f(t,y)$. At every specific point (t,y) we may not know any formula for the solution that goes through that point (it will exist by the aforementioned existence & uniqueness thm), but we do know its slope at that point – namely, whatever the value of f is at that point. We draw a little tangent vector of that slope at that point. We do this for a representative set of points in the plane. The result is the direction field of the ODE, which in principle gives us a good feel for the nature of the soln curves. Draw the direction field for the following examples:

1. $y' = y + 1$
2. $y' = y^2 + 1$
3. $y' = 1/y^2$
4. $y' = y + t.$

   Observe that Matlab does a far better job than we do of drawing a vector field.

c. **Numeric**. We use a numeric method to generate estimated values of the solution curve at selected points. Then we join them together to get a numerical approximation to the solution curve. Give a very rough 5-minute explanation of the Euler Method and carry it out for the example $y' = t^2 + y^2$, $y(0) = 1$, which has no elementary formula answer. Use Euler with one step to approximate $y(1)$.

d. **Qualitative**. This refers to a potpourri of techniques that we employ when none of the previous works so well. Here is a sample. Consider $y' = 1 + y^2$, $y(0) = 1$. Actually, we will have algebraic techniques to deal with this one, but here is a qualitative method. Observe that:

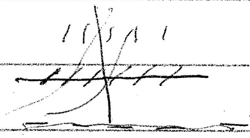
$$y^2 \le 1 + y^2 \le 2y^2, \text{ for } y \ge 1.$$

Therefore the solution to the IVP must lie between the sols to the two IVPs:
$y'=y^2$, $y(0)=1$ and $y'=2y^2$, $y(0)=1$. The latter two are easy to solve and yield:
$y = 1/(1 - t)$, and $y = 1/(1 - 2t)$, respectively. Draw the two curves. They blow up, at $t = 1$ and $t = 1/2$, resp. So the soln we desire must blow up in between the two. In fact the actual soln is $y = \tan(t + \pi/4)$, which blows up at $\pi/4$, in between ½ and 1.
Exc. Apply a qual technique to the example in **c** above and conclude that the soln curve blows up before $t = 1$. Thus the computation made there is specious. We will examine the reliability of numerical methods more closely when we consider the stability of diff eqns.

Vector Fields Drawn in Class

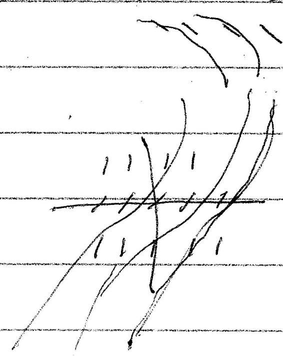
Sols

1. $y' = y + 1$



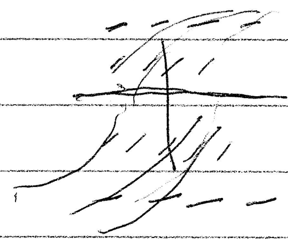
$y = ke^{t} - 1$

2. $y' = y^2 + 1$



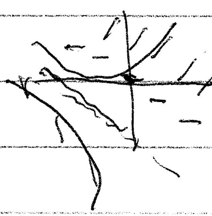
$y = \tan(t + c)$

3. $y' = \frac{1}{y^2}$



$y = (3t + c)^{1/3}$

4. $y' = y + t$



$y = ke^{t} - 1 - t$